J. Differential Equations 249 (2010) 1799–1816

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Multiple existence of solutions for a nonhomogeneous elliptic problem with critical exponent on $\mathbb{R}^N$

N. Hirano [a], Wan Se Kim [b,*]

[a] Department of Mathematics, Faculty of Engineering, Yokohama National University, Tokiwadai, Hodogayaku, Yokohama, Japan
[b] Deparment of Mathematics, Research institute for Natural Sciences, Hanyang University, Seoul, Republic of Korea

ARTICLE INFO

Article history:
Received 12 July 2009
Revised 7 March 2010
Available online 21 July 2010

MSC:
35J65

Keywords:
Critical Sobolev
Semilinear elliptic problem
Sign changing solutions

ABSTRACT

Let $N \geqslant 3$, $2^* = 2N/(N-2)$. Our purpose in this paper is to consider the existence and multiplicity of solutions of problem

$$\begin{cases} -\Delta u + \alpha u = |u|^{2^*-2}u + f & \text{on } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

where $\alpha > 0$, $f \in L^{2^*/(2^*-1)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $f \geqslant 0$ and $f \not\equiv 0$.

1. Introduction

Let $N \geqslant 3$, $2^* = 2N/(N-2)$. In the present paper, we consider the multiple existence of solutions of problem

$$\begin{cases} -\Delta u + \alpha u = |u|^{p-2}u + \mu f & \text{on } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{P_f}$$

where $\alpha > 0$, $p = 2^*$, $\mu > 0$ and $f \in L^{2^*/(2^*-1)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $f \geqslant 0$ and $f \not\equiv 0$. In the subcritical case, i.e., $p \in (2, 2^*)$, the existence of positive solution of (P_f) was established by Zhu [14] for f satisfying a decay condition on $\mathbb{R}^N$ (cf. Hirano [7]). The multiplicity of positive solutions of problem (P_f) for the subcritical case was studied by Deng and Li [3]. (See also [4].) In [8], the existence of three

* Corresponding author.
E-mail addresses: hira0918@ynu.ac.jp (N. Hirano), wanskim@hanyang.ac.kr (W.S. Kim).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.05.003

solutions of (P_f) was established. In the critical case $p = 2^*$, the problem is much more difficult than the subcritical case. The Palais–Smale condition does not hold at some critical levels and the effect of the nonhomogeneous term f to the multiple existence of solutions is delicate. The multiplicity of the solutions of (P_f) depends not only on the norm of f, but also the decay rate and the shape of f. In [2], it was shown that if $N < 6$ and $|x|^{N-2} f$ is bounded, then there exists $\mu^* > 0$ such that problem (P_f) has at least two positive solutions with $\mu \in (0, \mu^*)$. In case that $N \geqslant 6$, there exist $\mu^{**}, \mu_* > 0$ with $\mu_* < \mu^{**}$ such that for each $\mu \in (\mu^{**}, \mu^*)$, problem ($P_f$) possesses two positive solutions and for $\mu \in (0, \mu_*)$, problem (P_f) has a unique positive solution. The effect of the shape of f for the multiplicity of solutions of (P_f) was investigated in [9]. It was shown in [9] that if f has a compact support and has two peaks, then (P_f) has at least two positive solutions and two sign changing solutions.

In the present paper, we show the multiple existence of solutions of (P_f) for $3 \leqslant N \leqslant 5$. In the following, to indicate the dependence of α and μ, problem (P_f) is referred as problem ($P_{\alpha,\mu}$).

Our main result is the following.

Theorem 1.1. *Assume that $3 \leqslant N \leqslant 5$. If $|x|^{N-2} f$ is bounded, then there exist $\mu_* > 0$ and a function $\alpha : (0, \mu_*) \to \mathbb{R}^+$ such that for each $\mu \in (0, \mu_*)$ and $\alpha \in (0, \alpha(\mu))$, problem ($P_{\alpha,\mu}$) possesses at least three solutions.*

The main difficulty to prove Theorem 1.1 comes from the possible existence of two types of Palais–Smale sequence which do not contain any convergent subsequence. One type is a Palais–Smale sequence $\{u_n\}$ such that $\lim_{n\to\infty} \int_\Omega |u_n|^{2^*} \to 0$ on any bounded subset $\Omega \subset R^N$. The other one is a Palais–Smale sequence $\{u_n\}$ which concentrate in points in R^N. To overcome this difficulty, we need careful estimations for the critical levels of the functional associated with problem (P_f). In Section 2, we give notations and a few lemmas needed for our argument. In Section 3, we prove the existence of two positive solutions. In Section 4, we prove the existence of the third solution.

2. Preliminaries

Throughout the rest of this paper, we fix $f \in L^{2^*/(2^*-1)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. For $r > 0$, we denote by B_r the open ball in $\mathbb{R}^N$ centered at 0 with radius r. For $q > 1$, we denote by $|\cdot|_q$ the norm of $L^q(\mathbb{R}^N)$. We put $D^1(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N)\colon |\nabla u|_2 < \infty\}$. We also denote by $\|\cdot\|_0$ the norm of $D^1(\mathbb{R}^N)$ defined by $\|v\|_0^2 = \int_{\mathbb{R}^N} |\nabla v|^2\,dx$ for $v \in D^1(\mathbb{R}^N)$. We put $H = H^1(\mathbb{R}^N)$ and denote by $\|\cdot\|_\alpha$ the norm of $H^1(\mathbb{R}^N)$ defined by $\|u\|_\alpha^2 = |\nabla u|_2^2 + \alpha |u|_2^2$. For $u, v \in H^1(\mathbb{R}^N)$, we set $\langle u, v\rangle = \int_{\mathbb{R}^N} uv\,dx$. For each function $u : \mathbb{R}^N \to \mathbb{R}$, we put $u^+(x) = \max\{u(x), 0\}$ and $u^- = \max\{-u(x), 0\}$ on $\mathbb{R}^N$. For each Banach space E, $a \in \mathbb{R}$ and a functional $F : E \to \mathbb{R}$, we denote by F^a the level set $F^a = \{v \in H\colon F(v) \leqslant a\}$. We also put $F^{[a,b]} = \{v \in E\colon a \leqslant F(v) \leqslant b\}$. For each subset A of $D^1(\mathbb{R}^N)$ and $v \in D^1(\mathbb{R})$, $d_D(v, A)$ stands for the distance of v from the set A by the metric induced by the norm of $D^1(\mathbb{R}^N)$. For each $(\alpha, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $f \in L^{2^*/(2^*-1)}(\mathbb{R}^N)$, we define a functional $I_{\alpha,\mu}$ on H by

$$I_{\alpha,\mu}(u) = \frac{1}{2}\|u\|_\alpha^2 - \frac{1}{2^*}|u|_{2^*}^{2^*} - \mu\langle f, u\rangle \quad \text{for } u \in H.$$

Then the solutions of ($P_{\alpha,\mu}$) correspond to critical points of functional $I_{\alpha,\mu}$. We simply write I_0 instead of $I_{0,0}$. For $(x, \lambda) \in \mathbb{R}^N \times \mathbb{R}^+$, we define a function $U_{x,\lambda}$ by

$$U_{x,\lambda}(z) = m_1 \left[\frac{\lambda}{1 + \lambda^2 (z - x)^2}\right]^{\frac{N-2}{2}} \quad \text{for } z \in \mathbb{R}^N, \tag{2.1}$$

where $m_1 = (N(N-2))^{(N-2)/4}$. It is known that each $U_{x,\lambda}$ is a critical point of I_0. By the invariance of the norm of H under translation and the scaling

$$u \to u_R(x) = R^{-N/2^*} u(x/R), \quad R > 0,$$

we have that each $U_{x,\lambda}$ have the same critical value of I_0. We define an $N+1$ dimensional submanifold $\mathcal{M}$ of $D^1(\mathbb{R})$ by $\mathcal{M} = \{U_{x,\lambda}\colon (x,\lambda) \in \mathbb{R}^N \times \mathbb{R}^+\}$. For given $\varepsilon > 0$, we denote by $\mathcal{M}_\varepsilon$ a neighborhood of $\mathcal{M}_\varepsilon$ defined by

$$\mathcal{M}_\varepsilon = \{U_{x,\lambda} + D_\varepsilon(0)\colon (x,\lambda) \in \mathbb{R}^N \times \mathbb{R}^+\},$$

where $D_\varepsilon(0)$ is the open ball of $D^1(\mathbb{R}^N)$ centered at 0 with radius ε. For $(\alpha,\mu) \in \mathbb{R}^+ \times \mathbb{R}^+$, we set

$$\mathcal{S}^+_{\alpha,\mu} = \{v \in H\colon \|v\|^2_\alpha = |v|^{2^*}_{2^*} + \mu\langle f, v\rangle,\ I(v) > 0\}$$

and

$$\mathcal{S}^-_{\alpha,\mu} = \{v \in H\colon \|v\|^2_\lambda = |v|^{2^*}_{2^*} + \mu\langle f, v\rangle,\ I(v) < 0\}.$$

We simply write $\mathcal{S}_0$ instead of $\mathcal{S}^+_{0,0}$. It is easy to see that each critical point of problem $(\mathrm{P}_{\alpha,\mu})$ is contained in $\mathcal{S}^+_{\alpha,\mu} \cup \mathcal{S}^-_{\alpha,\mu}$, and

$$I_{\alpha,\mu}(u) = K_0\|u\|^2_\alpha - \frac{\mu(2^*-1)}{2^*}\langle u, f\rangle \quad \text{for each } u \in \mathcal{S}^+_{\alpha,\mu} \cup \mathcal{S}^-_{\alpha,\mu}, \tag{2.2}$$

where $K_0 = \frac{2^*-2}{2\cdot 2^*}$. Each function $U_{x,\lambda}$ satisfies

$$c_* = I_0(U_{x,\lambda}) = \min\{I_0(v)\colon v \in \mathcal{S}_0\}. \tag{2.3}$$

We also have

$$c_* = \inf\{I_{\alpha,0}(v)\colon v \in \mathcal{S}^+_{\alpha,0}\} \quad \text{for each } \alpha \in [0,1]. \tag{2.4}$$

It is known that there exists $\mu_0 > 0$ such that for each $u \in H\setminus\{0\}$ and $(\alpha,\mu) \in [0,1] \times [0,\mu_0)$, there exists a unique positive number $\tau > 0$ such that $\tau u \in \mathcal{S}^+_{\alpha,\mu}$ (cf. [8]). We also have

$$I_{\alpha,\mu}(\tau u) = \max\{I_{\alpha,\mu}(tu)\colon t \geqslant 0\}. \tag{2.5}$$

It is easy to see from the definition of $\mathcal{S}^+_{\alpha,\mu}$ that by choosing $\mu_0, l_0 > 0$ sufficiently small, we have

$$\inf\{\|v\|_0\colon v \in \mathcal{S}^+_{\alpha,\mu}\} > l_0 \quad \text{for all } (\alpha,\mu) \in [0,1] \times [0,\mu_0). \tag{2.6}$$

For each $(\alpha,\mu) \in [0,1] \times [0,\mu_0)$ and $u \in D^1(\mathbb{R}^N)$ with $u^\pm \not\equiv 0$, we put

$$\tau_{\alpha,\mu}(u) = \tau_1 u^+ - \tau_2 u^-,$$

where $\tau_1, \tau_2 > 0$ such that $\tau_1 u^+, \tau_2 u^- \in \mathcal{S}^+_{\alpha,\mu}$. Then we have $\tau_{\alpha,\mu}(u) \in \mathcal{S}^s_{\alpha,\mu}$. We next put

$$\mathcal{S}^s_{\alpha,\mu} = \{v \in H\colon v^\pm \in \mathcal{S}^\pm_{\alpha,\mu}\} \quad \text{and} \quad \mathcal{S}^s_{\alpha,\mu}(c) = \mathcal{S}^s_{\alpha,\mu} \cap I^c_{\alpha,\mu}$$

for $(\alpha,\mu) \in [0,1] \times (0,\mu_0)$ and $c > 0$.

One can see from the definition that $\mathcal{S}^s_{\alpha,\mu}$ is a submanifold of $\mathcal{S}^+_{\alpha,\mu}$ with codimension 1. Though the possible sign changing solution is contained in $\mathcal{S}^s_{\alpha,\mu}$, we cannot restrict ourselves on the manifold $\mathcal{S}^s_{\alpha,\mu}$ to find sign changing solutions, because $\mathcal{S}^s_{\alpha,\mu}$ is not necessarily C^1 manifold (cf. [1]). Then we need to work on a subset of $\mathcal{S}^+_{\alpha,\mu}$ containing of $\mathcal{S}^s_{\alpha,\mu}(c)$ for some $c>0$. For this purpose, we construct a kind of tuberian neighborhood of $\mathcal{S}^s_{\alpha,\mu}(c)$ in $\mathcal{S}^+_{\alpha,\mu}\cap I^c_{\alpha,\mu}$. The following lemma is a slight modification of Lemma 2.1 of [8].

Lemma 2.1. *(Cf. [8].) There exist $\mu_1\in(0,\mu_0)$ and $s_0\in(0,1)$ such that for each $(\alpha,\mu)\in[0,1]\times(0,\mu_1)$, there exist mappings $\sigma_\pm\in C([s_0,1]\times\mathcal{S}^s_{\alpha,\mu}(3c_*),[1,\infty))$ satisfying the following conditions:*

(1) *$\pm su^\pm\mp\sigma_\mp(s,u)u^\mp\in\mathcal{S}^+_{\alpha,\mu}$ for all $(s,u)\in[s_0,1]\times\mathcal{S}^s_{\alpha,\mu}(3c_*)$;*
(2) *for each $\sigma\in(s_0,1)$, there exist $m_{\sigma,1},m_{\sigma,2}>0$ such that*

$$m_{\sigma,1}<\frac{d}{ds}I_{\alpha,\mu}\big(\pm su^\pm\mp\sigma_\mp(s,u)u^\mp\big)<m_{\sigma,2}\quad \text{for all } (s,u)\in[s_0,\sigma]\times\mathcal{S}^s_{\alpha,\mu}(3c_*),$$

where $m_{\sigma,1},m_{\sigma,2}$ are independent of (α,μ).

Here we define a mapping $\sigma\in C([-1,1]\times\mathcal{S}^s_{\alpha,\mu}(3c_*),\mathcal{S}^+_{\alpha,\mu})$ by

$$\sigma_{\alpha,\mu}(s,u)=\begin{cases}(1+(1-s_0)s)u^+-\sigma_-(1+(1-s_0)s,u)u^-, & s\in(-1,0),\\ -(1-s(1-s_0))u^-+\sigma_+(1-s(1-s_0),u)u^+, & s\in[0,1]\end{cases}$$

for each $u\in\mathcal{S}^s_{\alpha,\mu}(3c_*)$. Then we have $\sigma_{\alpha,\mu}(0,u)=u$ for all $u\in\mathcal{S}^s_{\alpha,\mu}(3c_*)$. Moreover it follows from (1) and (2) of Lemma 2.1 that for each $u\in\mathcal{S}^s_{\alpha,\mu}(3c_*)$, the maximal of $I_{\alpha,\mu}(\sigma_{\alpha,\mu}(s,u))$ is attained at $u=\sigma_{\alpha,\mu}(0,u)$ and $I(\sigma_{\alpha,\mu}(s,u))$ decreases as $|s|$ increases. More precisely, we have that

$$I_{\alpha,\mu}\big(\sigma_{\alpha,\mu}(s,u)\big)\leqslant I_{\alpha,\mu}(u)-\theta(s)\quad\text{for } s\in[-1,1] \text{ and } u\in\mathcal{S}^s_{\alpha,\mu}(3c_*),\tag{2.7}$$

where $\theta(s)>0$ is independent of $(\alpha,\mu)\in[0,1]\times(0,\mu_0)$. We also have by (2) of Lemma 2.1 that there exist $m_1,m_2\in C([-1,1],\mathbb{R})$ such that $m_1(s)>0$ on $[-1,1]\backslash\{0\}$, and

$$m_1(s)\leqslant\left|\frac{d}{ds}I\big(\sigma_{\alpha,\mu}(s,u)\big)\right|\leqslant m_2(s)$$
$$\text{for } (\sigma,\mu)\in[0,1]\times[0,\mu_0],\ s\in[-1,1] \text{ and } u\in\mathcal{S}^s_{\alpha,\mu}(3c_*).\tag{2.8}$$

For each $c>0$ and $\delta\in[0,1]$, we put

$$\mathcal{C}_\delta\mathcal{S}^s_{\alpha,\mu}(c)=\big\{\sigma(s,u):\ |s|<\delta \text{ and } \mathcal{S}^s_{\alpha,\mu}(c)\big\}.$$

Then $\mathcal{C}_\delta\mathcal{S}^s_{\alpha,\mu}(c)$ is an open set in $\mathcal{S}^+_{\alpha,\mu}\cap I^c_{\alpha,\mu}$ containing $\mathcal{S}^s_{\alpha,\mu}(c)$. We define a mapping $\beta_{\alpha,\mu}\in C(\mathcal{C}_1\mathcal{S}^s_{\alpha,\mu}(3c),[-1,1])$ by

$$\beta_{\alpha,\mu}(v)=s\tag{2.9}$$

for $v=\sigma(s,w)$ with $s\in[-1,1]$, $w\in\mathcal{S}^s_{\alpha,\mu}$.

For simplicity of notations, we write $I,\mathcal{S}^s,\mathcal{S}^+$ and $\tau(\cdot)$ instead of $I_{\alpha,\mu},\mathcal{S}^s_{\alpha,\mu},\mathcal{S}^+_{\alpha,\mu}$ and $\tau_{\alpha,\mu}(\cdot)$, except for the case where (a,μ) should be specified clearly.

Lemma 2.2. *There exists $\varepsilon_0 \in (0, l_0/2)$ such that $\mathcal{M}_{\varepsilon_0} \cap (-\mathcal{M}_{\varepsilon_0}) = \emptyset$ and the metric projections $P : \mathcal{M}_{\varepsilon_0} \cup (-\mathcal{M}_{\varepsilon_0})$ $(\subset D^1(\mathbb{R}^N)) \to \mathcal{M} \cup (-\mathcal{M})$ is well defined, where l_0 is the constant defined in* (2.6).

Proof. By choosing $\varepsilon' > 0$ sufficiently small, we have that $\mathcal{M}_{\varepsilon'} \cap (-\mathcal{M}_{\varepsilon'}) = \emptyset$. We recall that for each $(x, \lambda) \in \mathbb{R}^N \times \mathbb{R}^+$,

$$\int_{R^N} \left| \nabla \frac{\partial U_{x,\lambda}}{\partial x_i} \right|^2 = K_1 \lambda^2, \qquad \int_{R^N} \left| \nabla \frac{\partial U_{x,\lambda}}{\partial \lambda} \right|^2 = \frac{K_2}{\lambda^2},$$

$$\int_{R^N} \left| \nabla \frac{\partial^2 U_{x,\lambda}}{\partial x_i \partial x_j} \right|^2 = K_3 \lambda^4, \qquad \left| \nabla \frac{\partial^2 U_{x,\lambda}}{\partial \lambda^2} \right|^2 = \frac{K_4}{\lambda^4} \quad \text{and} \quad \left| \nabla \frac{\partial^2 U_{x,\lambda}}{\partial x_i \partial \lambda} \right|^2 = K_5,$$

where $K_1, K_2, \ldots, K_5$ are positive constants independent of (x, λ) (cf. [10]). Then one can see that the principle curvature of $\mathcal{M} \subset D^1(\mathbb{R}^N)$ is bounded. Therefore there exists $\varepsilon_0 \in (0, \varepsilon')$ satisfying the assertion. $\square$

Lemma 2.3. *There exist $\varepsilon_1 > 0$ and $\mu_2 \in (0, \mu_1)$ such that*

$$\mathcal{S}^+ \cap I^{c_* + \varepsilon_1} \subset \mathcal{M}_{\varepsilon_0} \cup (-\mathcal{M}_{\varepsilon_0}) \quad \textit{for each } (\alpha, \mu) \in [0, 1] \times (0, \mu_2).$$

Proof. We recall that $c_* = \inf\{I_0(u)\colon u \in \mathcal{S}_0\}$ and c_* is attained only on $\mathcal{M} \cup (-\mathcal{M})$. It then follows that $u \in \mathcal{S}_0, I_0(u) \to c$ implies that $\mathrm{d}_D(u, \mathcal{M} \cup (-\mathcal{M})) \to 0$ (cf. [5]). Let $\{(\alpha_n, \mu_n)\} \subset [0, 1] \times (0, \mu_1)$ and $\{u_n\} \subset \mathcal{S}^+_{\alpha_n, \mu_n}$ be sequences such that $\lim_{n\to\infty} I_{\alpha_n, \mu_n}(u_n) = c_*$. From the definition of $\mathcal{S}^+_{\alpha,\mu}$,

$$|\nabla u_n|_2^2 \leqslant |\nabla u_n|_2^2 + \alpha_n |u_n|_2^2 = |u_n|_{2^*}^{2^*} + \mu_n \langle f, u_n \rangle \quad \text{for } n \geqslant 1.$$

Then for each $n \geqslant 1$, there exists $t_n \in (0, 1]$ such that $|\nabla t_n u_n|_2^2 = |t_n u_n|_{2^*}^{2^*} + \mu_n \langle f, t_n u_n \rangle$, and then by (2.2),

$$\begin{aligned} c_* &\leqslant I_0(t_n u_n) \\ &= K_0 |\nabla t_n u_n|_2^2 \\ &\leqslant K_0 \big(|\nabla u_n|_2^2 + \alpha_n |u_n|_2^2 \big) \\ &= I_{\alpha_n, \mu_n}(u_n) + \frac{2^* - 1}{2^*} \mu_n \langle f, u_n \rangle. \end{aligned}$$

Then noting that $\mu_n \to 0$, we find that $\lim_{n\to\infty} I_0(t_n u_n) = c_*$. This implies that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} \alpha_n |u_n|_2^2 = 0$. That is $\lim_{n\to\infty} I_0(u_n) = c_*$. Then we have $\lim_{n\to\infty} \mathrm{d}_D(u_n, \mathcal{M} \cup (-\mathcal{M})) = 0$. Since $\{\alpha_n\} \subset [0, 1]$ and $\{u_n\} \subset \mathcal{S}^+_{\alpha_n, \mu_n}$ are arbitrary, the assertion follows. $\square$

Here we define a mapping $\gamma_{\alpha,\mu} : \mathcal{S}^+_{\alpha,\mu} \cap I^{c_* + \varepsilon_1} \to \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^+$ by

$$\gamma_{\alpha,\mu}(u) = \big(\gamma^{(1)}_{\alpha,\mu}(u), \gamma^{(2)}_{\alpha,\mu}(u) \big) = (x, \lambda) \quad \text{for each } u \in \mathcal{S}^+ \cap I^{c_* + \varepsilon_1},$$

where $(x, \lambda) \in \mathbb{R}^N \times \mathbb{R}^+$ such that $U_{x,\lambda} = Pu$.

We write simply γ, γ_1 and γ_2 instead of $\gamma_{\alpha,\mu}, \gamma^{(1)}_{\alpha,\mu}$ and $\gamma^{(2)}_{\alpha,\mu}$.

Lemma 2.4. *There exists $(\mu_3, \delta_0) \in (0, \mu_2) \times (0, 1)$ such that for each $(\alpha, \mu) \in [0, 1] \times (0, \mu_3)$,*

$$u^{\pm} \in \pm\mathcal{M}_{\varepsilon_0} \qquad \text{if } u \in \mathcal{C}_{\delta_0}\mathcal{S}^s \cap I^{2c_*}.$$

Proof. First we recall a well-known fact. That is if $\alpha \in [0,1]$ and $\{v_n\} \subset \mathcal{S}^s_{\alpha,0}$ such that $\lim_{n\to\infty} I_{\alpha,0}(v_n) = 2c_*$ Then there exist sequences $\{(x_n, \lambda_n)\}, \{(x'_n, \lambda'_n)\} \subset \mathbb{R}^N \times \mathbb{R}^+$ such that

$$D\big((x_n, \lambda_m), \big(x'_n, \lambda'_n\big)\big) = \big|x_n - x'_n\big| + \big|\lambda_n - \lambda'_n\big| + \left|\frac{1}{\lambda_n} - \frac{1}{\lambda'_n}\right| \to \infty, \quad \text{as } n \to \infty$$

and

$$\lim_{n\to\infty} \big\|u_n^+ - U_{x_n,\lambda_n}\big\|_0 = \lim_{n\to\infty} \big\|u_n^- - U_{x'_n,\lambda'_n}\big\|_0 = 0.$$

Moreover if $\alpha > 0$, then $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \lambda'_n = \infty$ (cf. Proposition 3.1 of [11], and [14]). Let $\{(\alpha_n, \mu_n, \delta_n)\} \subset [0, 1] \times (0, \mu_3) \times (0, 1)$ be a sequence such that $\lim_{n\to\infty} \alpha_n = \alpha$, $\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \delta_n = 0$. Suppose that there exists a sequence $\{u_n\} \subset H$ such that $u_n \in \mathcal{C}_{\delta_n}\mathcal{S}^s_{\alpha_n,\mu_n} \cap I^{2c_*}$ for $n \geqslant 1$. Then since

$$\mathrm{d}_D\big(u_n^{\pm}, S^+_{\alpha_n,\mu_n}\big) \to 0, \quad \text{as } n \to \infty,$$

we have, by the fact above, that there exist $\{(x_n, \lambda_n)\}, \{(x'_n, \lambda'_n)\} \subset \mathbb{R}^N \times \mathbb{R}^+$ such that

$$\lim_{n\to\infty} \big\|u_n - (U_{x_n,\lambda_n} - U_{x'_n,\lambda'_n})\big\|_0 = 0 \quad \text{and} \quad \lim_{n\to\infty} D\big((x_n, \lambda_n), \big(x'_n, \lambda'_n\big)\big) = \infty.$$

This implies that we can choose positive numbers μ_3, δ_0 so small that

$$u^{\pm} \in \pm\mathcal{M}_{\varepsilon_0} \quad \text{for all } u \in \mathcal{C}_{\delta_0}\mathcal{S}^s \cap I^{2c_*}. \qquad \square$$

Remark 2.1. By (2) of Lemma 2.1, we have that there exist functions $\zeta_1, \zeta_2 : (0, \delta_0) \to \mathbb{R}^+$ such that for each $(\alpha, \mu) \in [0, 1] \times [0, \mu_3]$,

$$I(v) - \zeta_1(\delta) < I\big(\sigma(\pm\delta v)\big) \leqslant I(v) - \zeta_2(\delta) \quad \text{for all } v \in \mathcal{S}^s \cap I^{2c_*} \text{ and } \delta \in (0, \delta_0). \tag{2.10}$$

We assume by choosing δ_0 sufficiently small that $\zeta_1(\delta_0) < c_*$.

Lemma 2.5. *There exist $\zeta_0 > 0$ and $\delta_* \in (0, \delta_0)$ such that if $(\alpha, \mu) \in [0, 1] \times (0, \mu_3)$ and $a, b \in \mathbb{R}^+$ satisfy*

$$b - a < \zeta_0, \qquad c_{\alpha,\mu,1} + c_* < a < b < c_{\alpha,\mu,0} + 2c_*$$

and

$$\inf\big\{\big\|\nabla I(v)\big\|\colon v \in I^{[a,b]} \cap S^+\big\} > 0,$$

then there exists a deformation retract $\rho \in C([0, 1] \times I^b \cap \mathcal{S}^+, I^b \cap \mathcal{S}^+)$ such that

$$\rho\big(1, I^b\big) = I^a, \qquad I\big(\rho(t, v)\big) \leqslant I\big(\rho(s, v)\big) \quad \text{for } 0 \leqslant s \leqslant t \leqslant 1 \text{ and } v \in I^b \cap \mathcal{S}^+, \tag{2.11}$$

and

$$\rho\big(t, \mathcal{C}_{\delta_*}\mathcal{S}^s(2c_*)\big) \subset \mathcal{C}_{\delta_0}\mathcal{S}^s(3c_*) \quad \textit{for } t \in [0,1]. \tag{2.12}$$

Proof. Fix $\delta_* \in (0, \delta_0)$ and put

$$d_0 = \inf\big\{\|v - w\|_0\colon (\alpha,\mu) \in [0,1] \times (0,\mu_3),\ v \in \mathcal{C}_{\delta_*}\mathcal{S}^s(3c_*),\ w \in \mathcal{S}^+ \backslash \mathcal{C}_{\delta_0}\mathcal{S}^s(3c_*)\big\}.$$

We have $d_0 > 0$ by the definition of $\sigma_{\alpha,\mu}$ and Lemma 2.1. Put $\mathcal{C} = \mathcal{C}_{\delta_0}\mathcal{S}^s(3c_*) \backslash \mathcal{C}_{\delta_*}\mathcal{S}^s(3c_*)$. Then by (2.8), we have that there exist $d_1, d_2 > 0$ such that

$$d_1 < \big\|\nabla I(v)\big\| < d_2 \quad \text{for all } v \in \mathcal{C}.$$

We put $\zeta_0 = \frac{d_0 d_1^2}{4d_1}$. By a standard argument, we can construct a semiflow φ on S^+ associated with I such that

$$I\big(\varphi(t,v)\big) \leqslant I\big(\varphi(s,v)\big) \quad \text{for all } 0 < s < t \text{ and } v \in I^{[a,b]},$$
$$\frac{d_1}{2} \leqslant \left\|\frac{d\varphi(t,v)}{dt}\right\| \leqslant 2d_2 \quad \text{and} \quad \left\langle \frac{d\varphi(t,v)}{dt}, \nabla I\big(\varphi(t,v)\big)\right\rangle \leqslant -\frac{d_1^2}{2}$$

for $(t,v) \in [0,\infty) \times H$ such that $\varphi(t,v) \in \mathcal{C}$.

Now let $v \in \mathcal{C}_{\delta_*}\mathcal{S}^s(2c_*)$ and suppose that there exists $t > 0$ such that $I(\varphi(t,v)) \geqslant a$ and $\varphi(t,v)$ is contained in the boundary of $\mathcal{C}_{\delta_0}\mathcal{S}^s(3c_*)$ in $\mathcal{S}^+$. That is $\varphi(t,v) = \sigma(s,w)$ for some $s \in [-1,1]$ and $w \in \mathcal{S}^s(3c_*)$, and $s \in \{-\delta_0, \delta_0\}$ or $I(w) = 3c_*$ holds. By Remark 2.1, we have that

$$I(w) \leqslant I\big(\sigma(\delta_0, w)\big) + \zeta_1 \leqslant I\big(\sigma(s,w)\big) + \zeta_1 \leqslant 2c_* + \zeta_1 < 3c_*.$$

Therefore we have $s \in \{-\delta_0, \delta_0\}$. We may assume that $s = \delta_0$. Let $t_0 \in [0,t)$ be such that $\varphi(t_0, v) \in \mathcal{C}_{\delta_*}\mathcal{S}^s_{\alpha,\mu}(3c_*)$ and $\varphi(\tau, v) \notin \mathcal{C}_{\delta_*}\mathcal{S}^s_{\alpha,\mu}(3c_*)$ for $\tau \in (t_0, t]$. Then we have

$$d_0 \leqslant \big\|\varphi(t,v) - \varphi(t_0,v)\big\| \leqslant \int_{t_0}^{t} \left\|\frac{d\varphi(t,v)}{dt}\right\| dt \leqslant 2(t - t_0)d_2$$

and then

$$I\big(\varphi(t,v)\big) - I\big(\varphi(t_0,v)\big) \leqslant \int_{t_0}^{t} \left\langle \frac{d\varphi(t,v)}{dt}, \nabla I\big(\varphi(t,v)\big)\right\rangle dt \leqslant -\frac{d_1^2}{2}(t - t_0) \leqslant -\frac{d_0 d_1^2}{4d_2}.$$

Since $I(\varphi(t_0,v)) - I(\varphi(t,v)) \leqslant b - a < \zeta_0 = \frac{d_0 d_1^2}{4d_1}$, this is a contradiction. That is if $v \in \mathcal{C}_{\delta_*}\mathcal{S}^s(2c_*)$, $\varphi(t,v)$ stays in $\mathcal{C}_{\delta_0}\mathcal{S}^s(3c_*)$ as long as $I(\varphi(t,v)) \geqslant a$. Here for each $v \in \mathcal{C}_{\delta_*}\mathcal{S}^s(2c_*)$, we set $t_v \geqslant 0$ such that $I(t_v v) = a$, and define a deformation retract ρ by

$$\rho(t,v) = \begin{cases} \varphi(t_v t, v), & (t,v) \in [0,1] \times I^{(a,b]}, \\ v, & (t,v) \in [0,1] \times I^a. \end{cases}$$

Then ρ satisfies the desired properties. $\square$

Lemma 2.6. *Let $\{\Psi_t\}_{0\leqslant t\leqslant 1}$ be a homotopy of mappings in $C(\overline{B}_1\times[-1,1],\mathbb{R}^{N+2})$ such that*

(i) $\Psi_0(\cdot,s)_1\in C(\overline{B}_1\backslash B_{1/2},\mathbb{R}^N\backslash B_1)$ *is homotopic to identity mapping* $i:\overline{B}_1\backslash B_{1/2}\to\mathbb{R}^N\backslash\{0\}$ *for each* $s\in[-1,1]$,
(ii) $\Psi_0(x,\pm1)_2=\pm1$ *for* $x\in\overline{B}_1$,
(iii) $\Psi_0(x,s)_3\leqslant-1$ *for* $x\in\overline{B}_{1/2}\times[-1,1]$,
(iv) $\Psi_t(x,s)=\Psi_0(x,s)$ *for* $(x,s,t)\in\partial(B_1\times[-1,1])\times[0,1]$,
(v) $\Psi_1(x,s)\notin\{(y,0,\lambda):\ y\in\mathbb{R}^N,\ \lambda\leqslant0\}$ *for* $(x,s)\in(\overline{B}_1\times[-1,1])$

where

$$\Psi_t(x,s)=\big(\Psi_t(x,s)_1,\Psi_t(x,s)_2,\Psi_t(x,s)_3\big)\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}.$$

Then there exists $(x,s,t)\in B_1\times(-1,1)\times(0,1)$ *such that* $\Psi_t(x,s)=(0,0,0)$.

From the definition of Ψ, it is easy to see that the assertion holds. Then we omit the proof.

3. Existence of positive solutions

In this section and next section, we prove the existence of three solutions. We first show the existence of a positive solution of problem $(\mathrm{P}_{\alpha,\mu})$ in $S^-_{\alpha,\mu}$ using the result due to [2]. Next, we show the existence of another positive solution of $(\mathrm{P}_{\alpha,\mu})$ in $S^+_{\alpha,\mu}$. The existence of the third solution is proved in Section 4 by contradiction. That is we assume that there exist exactly two positive solutions of $(\mathrm{P}_{\alpha,\mu})$, and seek for a solution, which changes the sign, by careful estimations of critical level.

Proposition 3.1. *There exists $\mu_4>0$ such that for each $(\alpha,\mu)\in[0,1]\times(0,\mu_4)$, there exists a unique positive solution $u_{\alpha,\mu,0}\in H$ of $(\mathrm{P}_{\alpha,\mu})$ such that*

$$I(u_{\alpha,\mu,0})=\min_{u\in S^-_{\alpha,\mu}}I(u)=c_{\alpha,\mu,0}<0. \tag{3.1}$$

Proof. Our argument based on the existence result in [2]. By Theorem 1 of [2], choosing $\mu'\in(0,\mu_3)$ sufficiently small, we obtain a positive solution $u_{1,\mu,0}$ of $(\mathrm{P}_{1,\mu})$ satisfying (3.1) for each $\mu\in(0,\mu')$. Fix $\mu\in(0,\mu')$. Then we have for each $\alpha\in(0,1)$,

$$-\Delta u_{1,\mu,0}+\alpha u_{1,\mu,0}-u_{1,\mu,0}^{2^*-1}-\mu f\leqslant0\quad\text{on }\Omega. \tag{3.2}$$

That is $u_{1,\mu,0}$ is a lower solution of $(\mathrm{P}_{\alpha,\mu})$ for all $\alpha\in[0,1)$. Let $m_0>0$ be a constant such that $\|v\|_0\geqslant m_0|v|_{2^*}$ on $D^1(\mathbb{R}^N)$. Let $a>0$ such that $\frac12-\frac{m_0^{2^*}a^{2^*-2}}{2^*}>0$ and choose $\mu_4\in(0,\mu')$ so small that $\frac a2-\frac{m_0^{2^*}a^{2^*-1}}{2^*}-\mu_4m_0^{-1}|f|_{2^*/(2^*-1)}>0$. Then we have that for $u\in D^1(\mathbb{R}^N)$ with $\|u\|_0=a$, and $\mu\in(0,\mu_4)$,

$$\begin{aligned}I(u)&\geqslant\frac12|\nabla u|_2^2-\frac1{2^*}|u|_{2^*}^{2^*}-\mu\langle f,u\rangle\\&\geqslant a\left(\frac a2-\frac{m_0^{2^*}a^{2^*-1}}{2^*}-\mu m_0^{-1}|f|_{2^*/(2^*-1)}\right)\\&>0.\end{aligned} \tag{3.3}$$

We also have

$$\inf\{I(v):\ \|v\|_0 \leqslant a,\ v \geqslant u_{1,\mu,0}\} \leqslant I(u_{1,\mu,0}) < 0.$$

Combining (3.2), (3.3) and the inequality above, one can see that for each $(\alpha,\mu) \in [0,1]\times(0,\mu_4)$, there exists a solution $u_{\alpha,\mu,0}$ of $(\mathrm{P}_{\alpha,\mu})$ satisfying (3.1) and $u_{\alpha,\mu,0} \geqslant u_{1,\mu,0}$. The uniqueness of the assertion $u_{\alpha,\mu,1}$ follows by the same observation as in Section 2 of [2]. $\square$

Remark 3.1. It is clear from the definition of $u_{\alpha,\mu,0}$ that

$$\|u_{\alpha,\mu,0}\|_0 \to 0, \quad \text{as } \mu \to 0 \text{ uniformly with respect to } \alpha \in [0,1].$$

Moreover noting that $f \in L^\infty(\mathbb{R}^N)$, we have that $\{u_{\alpha,\mu,0}\} \subset C^1(\mathbb{R}^N)$ by a standard regularity argument.

Lemma 3.1. *There exists $\mu_* \in (0,\mu_4)$ such that for each $(\alpha,\mu) \in [0,1]\times(0,\mu_*)$, there exists a solution $u_{\alpha,\mu,1} \in H$ of $(\mathrm{P}_{\alpha,\mu})$ such that*

$$0 < I(u_{\alpha,\mu,1}) = \min_{u \in S^+_{\alpha,\mu}} I(u) = c_{\alpha,\mu,1} < c_{\alpha,\mu,0} + c_*. \tag{3.4}$$

Moreover for each $\mu \in (0,\mu_*)$, there exists $l(\mu) > 0$ such that

$$c_{\alpha,\mu,1} + l(\mu) \leqslant c_{\alpha,\mu,0} + c_* \quad \textit{for all } \alpha \in [0,1]. \tag{3.5}$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^{\mathrm{N}}, [0,1])$ be a mapping such that $\varphi(x) = 0$ for $|x| \geqslant 1$ and $\varphi(x) = 1$ for $|x| \leqslant 1/2$. We put

$$\widetilde{U}_{x,\lambda,d}(z) = \varphi\left(\frac{z-x}{d}\right)U_{x,\lambda}(z) \quad \text{for each } z \in \mathbb{R}^N \text{ and } (x,\lambda,d) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^+. \tag{3.6}$$

For simplicity, we write $U_{x,\lambda}$ instead of $U_{x,\lambda,1}$. Then we have $\lim_{\lambda\to\infty} \|\widetilde{U}_{x,\lambda} - U_{x,\lambda}\|_0 = 0$ for $x \in \mathbb{R}^N$ and $\alpha \in [0,1]$. Let $(\alpha,\mu) \in [0,1]\times(0,\mu_4)$. Since functional I has the mountain pass structure by (3.3), to show the existence of a solution satisfying (3.4), it is sufficient to show that

$$\sup_{t\in\mathbb{R}} I(u_{\alpha,\mu,0} + t\widetilde{U}_{0,\lambda}) < c_{\alpha,\mu,0} + c_* \quad \text{for some } \lambda > 0$$

(cf. [2,12]). We follow the argument employed in [2]. Since $u_{1,\mu,0} \in C^1(\mathbb{R}^N)$ and positive, we have that there exist $\delta > 0$ and $\eta > 0$ such that

$$u_{1,\mu,0}(0) \geqslant \eta \quad \text{on } B_\delta(0). \tag{3.7}$$

We put $u_0 = u_{\alpha,\mu,0}$. Fix $\lambda > 0$ and let $t_\lambda \in \mathbb{R}$ such that

$$I(u_0 + t_\lambda \widetilde{U}_{0,\lambda}) = \max \pm\mathcal{M}_{\varepsilon'} \subset I^{3c_*/2}_{\alpha,\mu} \quad \text{for all } (\alpha,\mu) \in [0,1]\times(0,\mu_1).\{I(u_0 + t\widetilde{U}_{0,\lambda}):\ t \in \mathbb{R}\}.$$

Then t_λ satisfies $\frac{d}{dt} I(u_0 + t\widetilde{U}_{0,\lambda})|_{t=t_\lambda} = 0$, i.e.,

$$\left\langle -\Delta(u_0 + t_\lambda\widetilde{U}_{0,\lambda}) + \alpha(u_0 + t_\lambda\widetilde{U}_{0,\lambda}) - (u_0 + t_\lambda\widetilde{U}_{0,\lambda})^{2^*-1} - \mu f, \widetilde{U}_{0,\lambda}\right\rangle = 0.$$

That is

$$t_\lambda\big(|\nabla\widetilde{U}_{0,\lambda}|_2^2+\alpha|\widetilde{U}_{0,\lambda}|_2^2\big)=t_\lambda^{2^*-1}|\widetilde{U}_{0,\lambda}|_{2^*}^{2^*}+A, \tag{3.8}$$

where

$$0<A\leqslant C\left(t_\lambda^{2^*-2}\int\limits_{\mathbb{R}^N}u_0\widetilde{U}_{0,\lambda}^{2^*-1}+t_\lambda\int\limits_{R^N}u_0^{2^*-2}\widetilde{U}_{0,\lambda}^2\right) \tag{3.9}$$

where $C>0$ is a constant such that

$$(a+b)^{2^*-1}\leqslant a^{2^*-1}+b^{2^*-1}+C\big(a^{2^*-2}b+ab^{2^*-2}\big)\quad \text{for } a,b\geqslant 0.$$

On the other hand, from [5], we have

$$|\nabla\widetilde{U}_{0,\lambda}|_2^2=K_0^{-1}c_*+O\big(\lambda^{-(N-2)}\big), \tag{3.10}$$

$$|\widetilde{U}_{0,\lambda}|_{2^*}^{2^*}=K_0^{-1}c_*+O\big(\lambda^{-N^2/(N-1)}\big), \tag{3.11}$$

and

$$|\widetilde{U}_{0,\lambda}|_2^2=\begin{cases}K_6\lambda^{-2}+O(\lambda^{-(N-2)}), & N\geqslant 5,\\ K_6\lambda^{-2}|\log\lambda|+O(\lambda^{-(N-2)}), & N=4,\\ O(\lambda^{-1}), & N=3\end{cases} \tag{3.12}$$

where K_6 is a positive constant. Then since $A>0$, we have by (3.8), (3.10), (3.11) and (3.12) that there exists $C_2>0$ such that $t_\lambda<C_2$ for λ sufficiently large. On the other hand, recalling that $|u_0|_{2^*}\to 0$, as $\mu\to 0$, we have by (3.8) and (3.9) that there exists $\mu_*\in(0,\mu_4)$ such that for $\mu\in(0,\mu_*)$,

$$t_\lambda\big(|\nabla\widetilde{U}_{0,\lambda}|_2^2+\alpha|\widetilde{U}_{0,\lambda}|_2^2\big)\leqslant 2t_\lambda^{2^*-1}|\widetilde{U}_{0,\lambda}|_{2^*}^{2^*}.$$

Then again by (3.10), (3.11) and (3.12) for $\mu\in(0,\mu_*)$ there exists $C_1>0$ such that $C_1<t_\lambda$ for λ sufficiently large.

Then noting that $(a+b)^{2^*}\geqslant a^{2^*}+b^{2^*}+2^*ab^{2^*-1}$ for $a,b\geqslant 0$, we find by (3.8) that

$$\begin{aligned}I(u_0+t_0\widetilde{U}_{0,\lambda})&=\frac{1}{2}\big(\big|\nabla(u_0+t_0\widetilde{U}_{0,\lambda})\big|_2^2+\alpha|u_0+t_0\widetilde{U}_{0,\lambda}|_2^2\big)\\&\quad-\frac{1}{2^*}|u_0+t_0\widetilde{U}_{0,\lambda}|_{2^*}^{2^*}-\langle u_0+t_0\widetilde{U}_{0,\lambda},\mu f\rangle\\&\leqslant c_{\alpha,\mu,0}+\frac{t_0^2}{2}\big(|\nabla\widetilde{U}_{0,\lambda}|_2^2+\alpha|\widetilde{U}_{0,\lambda}|^2\big)-\frac{t_0^{2^*}}{2^*}|\widetilde{U}_{0,\lambda}|_{2^*}^{2^*}\\&\quad-t_\lambda\big\langle-\Delta u_0+\alpha u_0-u_0^{2^*-1}-\mu f,\widetilde{U}_{0,\lambda}\big\rangle-C_1^{2^*-1}\int\limits_{R^N}u_0\widetilde{U}_{0,\lambda}^{2^*-1}\\&\leqslant c_{\alpha,\mu,0}+c_*+\frac{\alpha C_2^2}{2}|\widetilde{U}_{0,\lambda}|_2^2-C_1^{2^*-1}\int\limits_{R^N}u_0\widetilde{U}_{0,\lambda}^{2^*-1}+O\big(\lambda^{-(N-2)}\big).\end{aligned} \tag{3.13}$$

Since

$$\int\limits_{R^N} u_0 \widetilde{U}_{0,\lambda}^{2^*-1} \geqslant \int\limits_{R^N} u_{1,\mu,0} \widetilde{U}_{0,\lambda}^{2^*-1} \geqslant \eta \int\limits_{B_\delta(0)} \widetilde{U}_{0,\lambda}^{2^*-1} = O\left(\lambda^{-\frac{N-2}{2}}\right), \tag{3.14}$$

we have from (3.12) and (3.13) that there exist $l(\mu) > 0$ and $\lambda_\mu > 0$ such that

$$\sup_{\tau \in \mathbb{R}} I(u_{\alpha,\mu,0} + \tau \widetilde{U}_{0,\lambda_\mu}) = I(u_0 + t_0 \widetilde{U}_{0,\lambda_\mu}) \leqslant c_{\alpha,\mu,0} + c_* - l(\mu), \tag{3.15}$$

where $l(\mu)$ is independent of $\alpha \in (0,1]$. By the inequality above, one can see that there exists a solution u_1 of $(\mathrm{P}_{\alpha,\mu})$ satisfying (3.4). The positivity of u_1 follows by a standard argument. The inequality (3.5) follows from (3.15) and the definition of $c_{\alpha,\mu,1}$. □

Lemma 3.2. *For each $\mu \in (0,\mu_*)$, there exist positive numbers $C(\mu)$, $M_1(\mu)$ and $M_2(\mu)$ such that for each $(\alpha,\mu) \in [0,1] \times (0,\mu_*)$ and each solution $u_{\alpha,\mu,1} \in H$ of $(\mathrm{P}_{\alpha,\mu})$ satisfying* (3.4),

$$(x,\lambda) = \gamma(u_{\alpha,\mu,1}) \in B_{C(\mu)} \times \left(M_1(\mu), M_2(\mu)\right) \tag{3.16}$$

holds. Moreover for given $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\int\limits_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |u_{\alpha,\mu,1}|^{2^*} < \varepsilon \quad \textit{for all } \alpha \in [0,1]. \tag{3.17}$$

Proof. Fix $\mu \in (0,\mu_*)$. Let $\{\alpha_n\} \subset (0,1]$ and $\{u_{\alpha_n,\mu,1}\} \subset D^1(\mathbb{R}^N)$ be sequences such that $\alpha = \lim_{n\to\infty} \alpha_n \in [0,1]$ and each $u_{\alpha_n,\mu,1}$ is a solution of $(\mathrm{P}_{\alpha_n,\mu})$ satisfying (3.4). Since $c_{\alpha_n,\mu,1} \leqslant c_{\alpha_n,\mu,0} + c_* < c_*$, we have by Lemma 2.3 that each $u_n = u_{\alpha_n,\mu,1}$ has the form

$$u_n = U_{x_n,\lambda_n} + v_n, \tag{3.18}$$

where $(x_n,\lambda_n) = \gamma(u_n)$ and $\|v_n\|_0 < \varepsilon_0$ for $n \geqslant 1$. Since $f \in L^\infty(\mathbb{R}^N)$, and $\{|u_n|_{2^*}\}$, $\{|\nabla u_n|_2\}$ are bounded, we can see by the bootstrap argument using the L^p regularity result (cf. Theorem 3.8 of [13], Theorem 9.11 of [6]) that $\{u_n\}$ is bounded in $C^1(\mathbb{R}^N)$. Then by the Sobolev embedding theorem, we may assume by subtracting subsequence that $v_n \to v$ strongly in $L^{2^*}_{loc}(\mathbb{R}^N)$. Suppose that $|x_n| = \infty$ or $\max\{|\lambda_n|, |\lambda_n|^{-1}\} = \infty$ holds. Then we have

$$\left\langle -\Delta v + av - |v|^{2^*-2} v - \mu f, \varphi \right\rangle = \lim_{n\to\infty} \left\langle -\Delta u_n + \alpha_n u_n - u_n^{2^*-1} - \mu f, \varphi \right\rangle = 0$$

for each $\varphi \in C_0^\infty(\mathbb{R}^N)$. Noting that $\|v\|_0 < \varepsilon_0 < l_0$, we find that $v \in \mathcal{S}_{\alpha,\mu}^-$ and v is a solution of $(\mathrm{P}_{\alpha,\mu})$. That is $v = u_{\alpha,\mu,0}$. Therefore $I(v) = c_{\alpha,\mu.0.}$ On the other hand, recalling that

$$|\nabla u_n|^2 + \alpha_n |u_n|^2 - |u_n|^{2^*} - \mu\langle f, u_n\rangle = 0 \quad \text{for each } n \geqslant 1, \tag{3.19}$$

we have, putting $w_n = U_{x_n,\lambda_n} + v_n - v$, that

$$\begin{aligned} &\lim_{n\to\infty} \left(|\nabla w_n|^2 + \alpha_n |w_n|^2 - |w_n|^{2^*} - \mu\langle f, w_n\rangle\right) \\ &\quad = \lim_{n\to\infty} \left(|\nabla w_n|^2 + \alpha_n |w_n|^2 - |w_n|^{2^*}\right) = 0. \end{aligned}$$

Since (2.4) holds, this yields that $\lim_{n\to\infty} I_{\alpha_n,\mu}(w_n) = \lim_{n\to\infty} I_{\alpha,0}(w_n) \geqslant c_*$. Therefore

$$\lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} I(w_n) + I(v) \geqslant c_* + I(v) \geqslant c_* + c_{\alpha,\mu,0}.$$

This contradicts to (3.5). Thus we have $\{|x_n|\}$, $\{\lambda_n\}$ and $\{\lambda_n^{-1}\}$ are bounded. This completes the proof of the first part. Since $\{\alpha_n\}$ is arbitrary, to show (3.17), it is sufficient to show that $v_n \to v$ strongly in $D^1(\mathbb{R}^N)$. We may assume that $(x_n, \lambda_n) \to (x, \lambda) \in \mathbb{R}^N \times (0, \infty)$. We put $u = U_{x,\lambda} + v$. Then u is a solution of $(\mathrm{P}_{\alpha,\mu})$ and then again by (3.19), we find

$$\lim_{n\to\infty}\left(\left|\nabla(v_n - v)\right|^2 + \alpha_n|v_n - v|^2 - |v_n - v|^{2^*}\right) = 0.$$

Since $|v_n - v|_{2^*} \leqslant m_0^{-1}\|v_n - v\|_0 < 2m_0^{-1}\varepsilon_0$ for $n \geqslant 1$ and ε_0 can be chosen sufficiently small, we have from the equality above that $\|v_n - v\|_0 \to 0$, as $n \to \infty$. Then the second assertion follows. □

4. Existence of the third solution

Lemma 4.1. *Let $(\alpha, \mu) \in (0, 1] \times (0, \mu_*)$. If there exists $\{u_n\} \subset \mathcal{S}^+$ such that*

$$\lim_{n\to\infty} \mathrm{d}_D\left(u_n, \mathcal{S}^s\right) = 0, \qquad \lim_{n\to\infty}\left\|\nabla I(u_n)\right\|_\alpha = 0$$

and

$$c_{\alpha,\mu,1} + c_* < \lim_{n\to\infty} I(u_n) < c_{\alpha,\mu,0} + 2c_*.$$

Then there exists a solution $u \in H$ of $(\mathrm{P}_{\alpha,\mu})$ such that $c_{\alpha,\mu,1} < I(u) \leqslant c_{\alpha,\mu,0} + 2c_*$.

Proof. Let $(\alpha, \mu) \in (0, 1] \times (0, \mu_*)$, and assume that $\{u_n\} \subset \mathcal{S}^+$ satisfies the assumption. By Lemma 2.4, we have that u_n has the form $u_n = U_{x_n,\lambda_n} - U_{x'_n,\lambda'_n} + v_n$ for each $n \geqslant 1$, where $(x_n, \lambda_n) = \gamma(u_n^+)$ and $(x'_n, \lambda'_n) = \gamma(-u_n^-)$ and $\|v_n\|_0 < 2\varepsilon_0$.

Case 1. $\{(x_n, \lambda_n, \lambda_n^{-1})\}, \{(x'_n, \lambda'_n, {\lambda_n}'^{-1})\}$ are bounded in $\mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^+$. One can see by subtracting subsequences that $(x_n, \lambda_n, x'_n, \lambda'_n) \to (x_+, \lambda_+, x_-, \lambda_-)$, $u_n \to u = U_{x_+,\lambda_+} - U_{x_-,\lambda_-} + v$ strongly in $L^2_{loc}(\mathbb{R}^N)$ with $\|v\|_0 < 2\varepsilon_0$, $u_n \to u$ weakly in $H^1(\mathbb{R}^N)$ and $|u_n|^{2^*-2}u_n \to |u|^{2^*-2}u$ weakly in $(L^{2^*}(\mathbb{R}^N))^*$. This implies u is a solution of $(\mathrm{P}_{\alpha,\mu})$. It is easy to verify that for each $v \in S^+$ with $v \leqslant 0$, $I(v) \geqslant c_*$. Then since $u^\pm \not\equiv 0$, we have

$$I(u) = I\left(u^+\right) + I\left(u^-\right) > c_{\alpha,\mu,1} + c_*.$$

Case 2. $\{(x_n, \lambda_n, \lambda_n^{-1})\} \subset \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^+$ is bounded and

$$\lim_{n\to\infty}\left\{D\left((x_n, \lambda_n), \left(x'_n, \lambda'_n\right)\right)\right\} = \infty. \tag{4.1}$$

By subtracting subsequences, we have that $(x_n, \lambda_n) \to (x_+, \lambda_+)$, $u_n \to u = U_{x_+,\lambda_+} + v$ strongly in $L^2_{loc}(\mathbb{R}^N)$ with $\|v\|_0 < 2\varepsilon_0$, $u_n \to u$ weakly in $H^1(\mathbb{R}^N)$ and $|u_n|^{2^*-2}u_n \to |u|^{2^*-2}u$ weakly in $(L^{2^*}(\mathbb{R}^N))^*$. This implies u is a solution of $(\mathrm{P}_{\alpha,\mu})$. If $I(u) > c_{\alpha,\mu,1}$, the assertion follows. Then assume that $I(u) = c_{\alpha,\mu,1}$. Since $\lim_{n\to\infty}\|\nabla I_{\alpha,\mu}(u_n - u)\|_\alpha = \lim_{n\to\infty}\|\nabla I_{\alpha,0}(u_n - u)\|_\alpha = 0$ and $\lim_{n\to\infty} I_{\alpha,\mu}(u_n - u) = \lim_{n\to\infty} I_{\alpha,0}(u_n - u)$, we obtain that $\lim_{n\to\infty} I_{\alpha,\mu}(u_n - u) = c_*$. But by the assumption, we have

$$c_* < \lim_{n\to\infty} I_{\alpha,\mu}(u_n) - I_{\alpha,\mu}(u) = \lim_{n\to\infty} I_{\alpha,\mu}(u_n - u).$$

This is a contradiction.

Case 3. $\{(x'_n, \lambda'_n, (\lambda'_n)^{-1}\} \subset \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^+$ is bounded and (4.1) holds. By the same argument as in Case 2, we reaches to a contradiction.

Case 4. $\{(x_n, \lambda_n, \lambda_n^{-1})\}, \{(x'_n, \lambda'_n, (\lambda'_n)^{-1}\} \subset \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^+$ are unbounded. By subtracting subsequences, we have that $u_n \to u$ strongly in $L^2_{loc}(\mathbb{R}^N)$ with $\|u\|_0 < 2\varepsilon_0$, $u_n \to u$ weakly in $H^1(\mathbb{R}^N)$ and $|u_n|^{2^*-2}u_n \to |u|^{2^*-2}u$ weakly in $(L^{2^*}(\mathbb{R}^N))^*$. Then by (2.6), $u \in \mathcal{S}^-_{\alpha,\mu}$ and u is a solution of $(\mathrm{P}_{\alpha,\mu})$, i.e. $u = u_{\alpha,\mu,0}$. We also have

$$\begin{aligned}\lim_{n\to\infty} I_{\alpha,\mu}(u_n) &= I_{\alpha,\mu}(u) + \lim_{n\to\infty} I_{\alpha,0}(U_{x_n,\lambda_n}) + \lim_{n\to\infty} I_{\alpha,0}(U_{x'_n,\lambda'_n}) \\ &= c_{\alpha,\mu,0} + 2c_*.\end{aligned}$$

This contradicts to the assumption and the proof is completed. □

Lemma 4.2. *Let* $(\alpha, \mu) \in (0, 1] \times (0, \mu_*)$. *Then*

(1) *For each* $\lambda > 0$, *there exists* $\varepsilon > 0$ *such that*

$$\gamma_2(u^-) \geqslant \lambda \quad \textit{for } u \in \mathcal{S}^s \cap I^{c_{\alpha,\mu,1}+c_*+\varepsilon}.$$

(2) *For each* $\lambda > 0$, *there exists* $\delta_{\alpha,\mu} > 0$ *such that*

$$\inf\{I(u):\ u \in \Gamma_{\alpha,\mu}\} > c_{\alpha,\mu,1} + c_* + \delta_{\alpha,\mu},$$

where

$$\Gamma_{\alpha,\mu} = \{u \in \mathcal{S}^s_{\alpha,\mu}:\ \gamma_2(u^{-1}) - \gamma_2(u^+) \leqslant 0\}.$$

Proof. (1) Let $(\alpha, \mu) \in (0, 1] \times (0, \mu_*)$. Let $\{u_n\} \subset \mathcal{S}^s$ be a sequence such that $\lim_{n\to\infty} I(u_n) = c_{\alpha,\mu,1} + c_*$ and $\{\gamma_2(u_n^-)\}$ is bounded. Then since $I(u_n^-) \geqslant c_*$, we have that $\lim_{n\to\infty} I(u_n^+) = c_{\alpha,\mu,1}$. That is $\{u_n^+\} \subset \mathcal{S}^+$ is a minimizing sequence for I in $\mathcal{S}^+$. Then each u_n^+ has the form (3.18) and repeating the argument in the proof of Lemma 3.2, we have that $u_n \to u^+ = u_{\alpha,\mu,1}$ strongly in H, where $u_{\alpha,\mu,1}$ is a solution of $(\mathrm{P}_{\alpha,\mu})$ satisfying (3.4). Then we have as in the proof of Lemma 4.1 that u_n has the form $u_n = u_{\alpha,\mu,1} - U_{x_n,\lambda_n} + v_n$ for each $n \geqslant 1$, where $(x_n, \lambda_n) \in \mathbb{R}^N \times (0, \infty)$ and $\|v_n\| < 2\varepsilon_0$ for n sufficiently large. On the other hand,

$$\lim_{n\to\infty} I_0(u_n^-) \leqslant \lim_{n\to\infty}\left(\frac{1}{2}\|u_n^-\|_\alpha^2 - \frac{1}{2^*}|u_n^-|_{2^*}^{2^*} - \mu\langle f, u_n^-\rangle\right) = \lim_{n\to\infty} I(u_n^-) = c_*.$$

Since

$$\|u_n^-\|_0^2 \leqslant \|u_n^-\|_\alpha^2 = |u_n^-|_{2^*}^{2^*} + \mu\langle f, u_n^-\rangle,$$

we have that there exists $t_n \in (0, 1)$ such that $t_n u_n^- \in \mathcal{S}^+_{0,\mu}$. Then we have

$$\lim_{n\to\infty} I_0\big(t_n u_n^-\big) = \lim_{n\to\infty} K_0\big\|t_n u_n^-\big\|_0^2$$
$$\leqslant \lim_{n\to\infty} K_0\big|\nabla u_n^-\big|_2^2$$
$$\leqslant \lim_{n\to\infty}\big(K_0\big(\big|\nabla u_n^-\big|_2^2+\alpha\big|u_n^-\big|_2^2\big)-\mu\big\langle f,u_n^-\big\rangle\big)\leqslant c_*.$$

Since $\lim_{n\to\infty} I_0(t_n u_n^-)\geqslant c_*$ and $\langle f,u_n^-\rangle\leqslant 0$ for $n\geqslant 1$, this implies that $\lim_{n\to\infty} t_n=1$ and $\lim_{n\to\infty}|u_n^-|_2^2=0$. Therefore we find that $\lim_{n\to\infty} I_0(u_n^-)=c_*$ and each u_n^- has the form $u_n^-=-U_{x_n,\lambda_n}+w_n$ with $(x_n,\lambda_n)\in\mathbb{R}^N\times\mathbb{R}^+$ and $\|w_n\|_0<\varepsilon_0$. Since $\{\lambda_n\}$ is bounded, we have that

$$\inf_{n\geqslant 1}\int\limits_{R^N}\big|u_n^-\big|^2=\inf_{n\geqslant 1}\int\limits_{R^N}|-U_{x_n,\lambda_n}+w_n|^2>0.$$

This implies that

$$\liminf I\big(u_n^-\big)=\lim_{n\to\infty} I_0\big(u_n^-\big)+\liminf_{n\to\infty}\int\limits_{R^N}\big|u_n^-\big|^2>c_*.$$

This is a contradiction. Then the assertion holds.

(2) Let $(\alpha,\mu)\in(0,1]\times(0,\mu_*)$. Suppose contrary that there exists a sequence $\{u_n\}\subset\Gamma_{\alpha,\mu}$ such that

$$\lim_{n\to\infty} I_{\alpha,\mu}(u_n)=c_{\alpha,\mu,1}+c_*.$$

Then as above, $u_n^+\to u_{\alpha,\mu,1}$ strongly in $D^1(\mathbb{R}^N)$. By (1), we have that there exists $\varepsilon>0$ such that for each $u\in\mathcal{S}^s$ such that $I(u)\leqslant c_{\alpha,\mu,1}+c_*+\varepsilon$, $\gamma_2(u^-)\geqslant 2M_2(\mu)$. Then since $\lim_{n\to\infty}\gamma_2(u_n^+)=\lim_{n\to\infty}\gamma_2(v)\leqslant M_2(\mu)$ by (3.16) and $\liminf_{n\to\infty}\gamma_2(u_n^-)\geqslant 2M_2(\mu)$, we have that $u_n\notin\Gamma_{\alpha,\mu}$ for n sufficiently large. This contradicts the assumption. □

Lemma 4.3. *For each $\mu\in(0,\mu_*)$, there exist $\alpha(\mu)>0$, $\lambda_0=\lambda_0(\mu)\in(0,M_1(\mu)/4)$, $R_*>0$ and $d_0=d_0(\mu)>0$ such that for $\alpha\in(0,\alpha(\mu))$,*

$$I\big(\tau(u_{\alpha,\mu,1}-\widetilde{U}_{x,\lambda_0,d_0})\big)<c_{\alpha,\mu,1}+c_*+\min\bigg\{l(\mu),\frac{\zeta_2(\delta_*)}{2},\zeta_0\bigg\}\quad\text{for all } x\in\mathbb{R}^N,\tag{4.2}$$

$$I\big(\tau(u_{\alpha,\mu,1}-\widetilde{U}_{x,\lambda,d_0})\big)<c_{\alpha,\mu,1}+c_*+\min\bigg\{l(\mu),\frac{\zeta_2(\delta_*)}{2},\zeta_0\bigg\}$$
$$\text{for all } (x,\lambda)\in\big(\mathbb{R}^N\backslash B_{R_*}\big)\times[\lambda_0,\infty),\tag{4.3}$$

$$|x_{\alpha,\mu,1}-x_+|<1,\qquad|\lambda_{\alpha,\mu,1}-\lambda_+|<\frac{M_1(\mu)}{4}\tag{4.4}$$

and

$$|x-x_-|<1,\qquad|\lambda_0-\lambda_-|<\frac{M_1(\mu)}{4}\tag{4.5}$$

where ζ_0, ζ_2 are constants defined in Remark 2.1 *and Lemma* 2.5*,*

$$(x_{\alpha,\mu,1}, \lambda_{\alpha,\mu,1}) = \gamma(u_{\alpha,\mu,1}),$$
$$(x_+, \lambda_+) = \gamma\big(\tau(u_{\alpha,\mu,1} - \widetilde{U}_{x,\lambda_0,d_0})^+\big) \quad \textit{and} \quad (x_-, \lambda_-) = \gamma\big(\tau(u_{\alpha,\mu,1} - \tau\widetilde{U}_{x,\lambda_0,d_0})^-\big).$$

Proof. Let $\mu \in (0, \mu_*)$ and put $c(\mu) = \min\{l(\mu), \frac{\zeta_2(\delta_*)}{2}, \zeta_0\}$. For each $\alpha \in (0, 1]$, let $u_{\alpha,\mu,1} \in H$ be a solution of $(\mathrm{P}_{\alpha,\mu})$ satisfying (3.4). We put $u_{x,\lambda} = u_{\alpha,\mu,1} - U_{x,\lambda}$ for $(x, \lambda) \in \mathbb{R}^N \times \mathbb{R}^+$ and $\alpha \in (0, 1]$. Then we have from the definition of $U_{x,\lambda}$ that

$$\lim_{\lambda\to 0} I_{0,\mu}\big(\tau_{0,\mu}(u_{x,\lambda})\big) = \lim_{\lambda\to 0} I_{0,\mu}(u_{x,\lambda}) = \lim_{\lambda\to 0} I_{0,\mu}(u_{\alpha,\mu,1}) + \lim_{\lambda\to 0} I_0(U_{x,\lambda}) \leqslant c_{\alpha,\mu,1} + c_*.$$

We also have that for each $a > 0$,

$$\lim_{|x|\to\infty} I_{0,\mu}\big(\tau_{0,\mu}(u_{x,\lambda})\big) = \lim_{|x|\to\infty} I_{0,\mu}(u_{x,\lambda}) = \lim_{|x|\to\infty} I_{0,\mu}(u_{\alpha,\mu,1}) + \lim_{|x|\to\infty} I_0(U_{x,\lambda}) \leqslant c_{\alpha,\mu,1} + c_*$$

uniformly for $\lambda \in [a, \infty)$. Then we can choose $\lambda_0 > 0$ and $R_* > 0$ such that

$$I_{0,\mu}\big(\tau_{0,\mu}(u_{x,\lambda_0})\big) \leqslant c_{\alpha,\mu,1} + c_* + c(\mu)/3 \quad \text{for all } x \in \mathbb{R}^N$$

and

$$I_{0,\mu}\big(\tau_{0,\mu}(u_{x,\lambda})\big) \leqslant c_{\alpha,\mu,1} + c_* + c(\mu)/3 \quad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geqslant R_* \text{ and } \lambda \geqslant \lambda_0.$$

Then since

$$\lim_{d\to\infty} \|U_{x,\lambda} - \widetilde{U}_{x,\lambda,d}\|_0 = 0, \quad \text{uniformly for } (x, \lambda) \in \mathbb{R}^N \times [\lambda_0, \infty), \tag{4.6}$$

we can choose $d_0 = d_0(\mu)$ sufficiently large that

$$I_{0,\mu}\big(\tau_{0,\mu}(\widetilde{u}_{x,\lambda_0})\big) \leqslant c_{\alpha,\mu,1} + c_* + 2c(\mu)/3 \quad \text{for all } x \in \mathbb{R}^N \tag{4.7}$$

and

$$I_{0,\mu}\big(\tau_{0,\mu}(\widetilde{u}_{x,\lambda})\big) \leqslant c_{\alpha,\mu,1} + c_* + 2c(\mu)/3 \quad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geqslant R_* \text{ and } \lambda \geqslant \lambda_0, \tag{4.8}$$

where $\widetilde{u}_{x,\lambda} = u_{\alpha,\mu,1} - \widetilde{U}_{x,\lambda,d_0}$ for $(x, \lambda) \in \mathbb{R}^N \times [\lambda_0, \infty)$. Then since $I(\widetilde{u}_{x,\lambda}) = I_{0,\mu}(\widetilde{u}_{x,\lambda}) + \alpha|\widetilde{u}_{x,\lambda}|_2^2$ for $(x, \lambda) \in \mathbb{R}^N \times \mathbb{R}^+$, we can choose $\alpha(\mu) > 0$ so small that

$$I\big(\tau_{\alpha,\mu}(\widetilde{u}_{x,\lambda})\big) < I_{0,\mu}\big(\tau_{0,\mu}(\widetilde{u}_{x,\mu})\big) + c(\mu)/3$$
$$\text{for all } \alpha \in \big(0, \alpha(\mu)\big) \text{ and } (x, \lambda) \in \mathbb{R}^N \times [\lambda_0, \infty). \tag{4.9}$$

Therefore by (4.7), (4.8) and (4.9), (4.2) and (4.3) follow. On the other hand, noting that (3.16) and (3.17) hold, we find that

$$\lim_{\lambda\to 0}\big\|u^+_{x,\lambda} - u_{\alpha,\mu,1}\big\|_0 = \lim_{\lambda\to 0}\big\|u^-_{x,\lambda} - U_{x,\lambda}\big\|_0 = 0 \quad \text{uniformly for } x \in \mathbb{R}^N \text{ and } \alpha \in [0, 1]. \tag{4.10}$$

Then combining (4.10) with (4.6), we find that by assuming λ_0 and d_0 sufficiently large (4.4) and (4.5) hold. □

Proof of Theorem 1.1. Let $\mu \in (0, \mu_*)$ and fix $\alpha \in (0, \alpha(\mu))$. We assume that there is no critical point of I with critical value different from $c_{\alpha,\mu,1}$ and $c_{\alpha,\mu,0}$. By (1) of Lemma 4.2, we have that for given $\lambda_1 > 2M_2(\mu) + 1$, there exists $\varepsilon > 0$ such that

$$\gamma_1\big(u^-\big) \geqslant \lambda_1 \quad \text{for } u \in \mathcal{S}^s \cap I^{c_{\alpha,\mu,1}+c_*+\varepsilon}. \tag{4.11}$$

Since $c_{\alpha,\mu,1} + c_* + l(\mu) < c_{\alpha,\mu,0} + 2c_*$, we have by Lemma 4.1,

$$\inf\big\{\big\|\nabla I(v)\big\|:\ v \in \mathcal{S}^+,\ a < I(v) < b\big\} > 0, \tag{4.12}$$

where

$$\begin{aligned} a &= c_{\alpha,\mu,1} + c_* + \min\left\{\frac{l(\mu)}{2}, \frac{\zeta_2(\delta_*)}{4}, \frac{\zeta_0}{2}, \delta_{\alpha,\mu}, \varepsilon\right\}, \\ b &= c_{\alpha,\mu,1} + c_* + \min\left\{l(\mu), \frac{\zeta_2(\delta_*)}{2}, \zeta_0\right\}. \end{aligned} \tag{4.13}$$

Then since $b - a < \zeta_0$, we have by Lemma 2.5 that there exists a deformation retract ρ satisfying (2.11) and (2.12). Let $u_{\alpha,\mu,1} \in \mathcal{S}^+$ be a solution of $(\mathrm{P}_{\alpha,\mu})$ satisfying (3.4). Then by Lemma 4.3, we have that for $\lambda_0 = \lambda_0(\mu) \in (0, M_1(\mu)/4]$,

$$I\big(\tau(u_{\alpha,\mu,1} - \widetilde{U}_{x,\lambda_0,d_0})\big) < c_{\alpha,\mu,1} + c_* + \min\left\{l(\mu), \frac{\zeta_2(\delta_*)}{2}, \zeta_0\right\} = b \quad \text{for all } x \in \mathbb{R}^N.$$

On the other hand, we can choose $\lambda_2 > \lambda_1$ so large that

$$I\big(\tau(u_{\alpha,\mu,1} - \widetilde{U}_{x,\lambda_2})\big) \leqslant a \quad \text{for all } x \in \mathbb{R}^N. \tag{4.14}$$

Let $R_* > 0$ be the number obtained in Lemma 4.3. We may assume that $R_* - C(\mu) - 2 > 1$. Here we put

$$u_{\alpha,\mu,x} = \begin{cases} \tau(u_{\alpha,\mu,1} - \widetilde{U}_{2R_*x,\lambda_0,d_0}) & \text{for } x \in B_{1/2}, \\ \tau(u_{\alpha,\mu,1} - \widetilde{U}_{R_*\frac{x}{|x|},\lambda_0+(\lambda_2-\lambda_0)(2|x|-1),d_0}) & \text{for } x \in \overline{B}_1 \backslash B_{1/2}. \end{cases}$$

By Lemma 4.3, we find that $I(u_{\alpha,\mu,x}) \leqslant b$ for all $x \in \overline{B}_1$. Then using the function $u_{\alpha,\mu,x}$, we can define functions $\varphi_t : \overline{B}_1 \times [-1, 1] \to \mathcal{S}^+$ for $t \in [0, 1]$ by

$$\varphi_0(x, s) = \sigma(\delta_* s, u_{\alpha,\mu,x}) \quad \text{for } (x, s) \in \overline{B}_1 \times [-1, 1],$$

and

$$\varphi_t(x, s) = \rho\big(t, \varphi_0(x, s)\big) \quad \text{for } (x, s, t) \in \overline{B}_1 \times [-1, 1] \times [0, 1].$$

Then by Lemma 2.5, we have that $\varphi_t(x, s) \in \mathcal{C}_{\delta_0}\mathcal{S}^s(3c_*)$ on $\overline{B}_1 \times [-1, 1] \times [0, 1]$. We next define a homotopy $\{\Psi_t\}_{0 \leqslant t \leqslant 1}$ of mapping from $\overline{B}_1 \times [-1, 1] \to \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ by

$$\Psi_t(x,s) = \big(\Psi_t(x,s)_1, \Psi_t(x,s)_2, \Psi_t(x,s)_3\big),$$

where

$$\Psi_t(x,s)_1 = \gamma_1\big(\varphi_t(x,s)^-\big) - \gamma_1\big(\varphi_t(x,s)^+\big),$$
$$\Psi_t(x,s)_2 = \beta\big(\varphi_t(x,s)\big),$$
$$\Psi_t(x,s)_3 = \frac{4}{M_1(\mu)}\big(\gamma_2\big(\varphi_t(x,s)^-\big) - \gamma_2\big(\varphi_t(x,s)^+\big)\big)$$

for each $(x,s) \in \bar{B}_1 \times [-1,1]$, where $\beta = \beta_{\alpha,\mu}$ is the mapping defined by (2.9). Putting $(x_+, \lambda_+) = \gamma(u^+_{\alpha,\mu,x})$ and $(x_-, \lambda_-) = \gamma(u_{\alpha,\mu,x}{}^-)$, we have

$$\Psi_0(x,s)_1 = x_- - x_+ = \frac{R_* x}{|x|} - x_{\alpha,\mu,1} + \left(x_- - \frac{R_* x}{|x|}\right) + (x_{\alpha,\mu,1} - x_+)$$

for $(x,s) \in (\bar{B}_1 \backslash B_{1/2}) \times [-1,1]$. Then by (3.16), (4.4) and (4.5) that

$$\big|\Psi_0(x,s)_1\big| \geqslant R_* - C(\mu) - 2 \geqslant 1$$

and one can see that the mapping $\bar{B}_1 \backslash B_{1/2} \ni x \to \Psi_0(x,s)_1$ is homotopic to identity mapping $i : \bar{B}_1 \backslash B_{1/2} \to \mathbb{R}^N \backslash \{0\}$. On the other hand, $\Psi_0(x,\pm 1)_2 = \pm 1$ for $x \in \bar{B}_1$ by the definition of $\varphi_0(x,s)$ and β. That is (i) and (ii) of Lemma 2.6 hold. By (3.16), $\gamma_1(u_{\alpha,\mu,1}) \geqslant M_1(\mu)$. Then noting that $\lambda_0 \leqslant M_1(\mu)/4$, we have by (4.4) and (4.5) that

$$\lambda_- - \lambda_+ \leqslant \lambda_0 + |\lambda_- - \lambda_0| - \lambda_{\alpha,\mu,1} + |\lambda_{\alpha,\mu,1} - \lambda_+| \leqslant -\frac{M_1(\mu)}{4}.$$

Then by the definition of $u_{\alpha,\mu,x}$, we have $\Psi_0(x,s)_3 \leqslant -1$ on $B_{1/2}(0) \times [-1,1]$. That is (iii) of Lemma 2.6 holds. Next we will see that (iv) holds. From the definition of $u_{\alpha,\mu,x}$,

$$I\big(\varphi_0(x,s)\big) \leqslant I\big(\varphi_0(x,0)\big) = I(u_{\alpha,\mu,x}) \leqslant a \quad \text{for } (x,s) \in \partial B_1 \times [-1,1].$$

We have by (2.10) and the definition of a and b that

$$I\big(\varphi_0(x,\pm 1)\big) = I\big(\sigma(\pm\delta_*, u_{\alpha,\mu,x})\big) \leqslant I(u_{\alpha,\mu,x}) - \zeta_2(\delta_*) \leqslant a \quad \text{for } x \in \bar{B}_1.$$

Therefore (iv) of Lemma 2.6 holds. Lastly we see that (v) of Lemma 2.6 holds. Since $I(\Psi_1(x,s)) \leqslant a$, we have by (2) of Lemma 4.2 that for each $\varphi_1(x,s) \in \mathcal{S}^s$,

$$\Psi_1(x,s)_3 = \gamma_1\big(\varphi_t(x,s)^-\big) - \gamma_1\big(\varphi_t(x,s)^+\big) > 0.$$

Therefore (v) holds. Then by Lemma 2.6, there exists $(x,s,t) \in B_1 \times (-1,1) \times (0,1]$ such that $\Psi_t(x,s) = (0,0,0)$. By Lemma 2.4, $\varphi_t(x,s)$ has the form $\varphi_t(x,s) = U_{x_+,\lambda_+} - U_{x_-,\lambda_-} + v$, where $\|v\|_0 < 2\varepsilon_0$. Since $\Psi_t(x,s) = (0,0,0)$, we have $(x_+,\lambda_+) = (x_-,\lambda_-)$. Then $\varphi_t(x,s) = v$. Since $2\varepsilon_0 < l_0$, we find by (2.6) that $\varphi_t(x,s) \notin \mathcal{S}^+$. This is a contradiction. Then the assertion follows. □

References

[1] A. Castro, M. Clapp, The effect of the domain topology on the number of minimal nodal solutions of an elliptic equation at critical growth in a symmetric domain, Nonlinearity 16 (2003) 579–590.

[2] Y. Deng, Y. Li, Existence and bifurcation of the positive solutions for a semilinear equation with critical exponent, J. Differential Equations 130 (1996) 179–200.

[3] Y. Deng, Y. Li, Existence of multiple positive solutions for a semilinear elliptic equation, Adv. Differential Equations 2 (1997) 361–382.

[4] G. Djairo, D.G. de Figueiredo, On superlinear Ambrosetti–Prodi problem on contractible domains, Nonlinear Anal. 8 (1984) 655–665.

[5] G. Cerami, S. Solimini, M. Struwe, Some existence results for superlinear elliptic boundary value problems involving critical exponents, J. Funct. Anal. 69 (1986) 289–306.

[6] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, 1998.

[7] N. Hirano, Existence of entire positive solutions for non homogeneous elliptic equations, Nonlinear Anal. 29 (1997) 889–901.

[8] N. Hirano, Multiple existence of solutions for a nonhomogeneous elliptic problem on R^N, J. Math. Anal. Appl. 336 (2007) 506–522.

[9] N. Hirano, A. Micheletti, A. Pistoia, Existence of non-positive solutions for a nonhomogeneous elliptic problems on R^N, J. Nonlinear Convex Anal. 9 (2008) 309–329.

[10] O. Rey, Concentration of solutions to elliptic equations with critical nonlinearity, Ann. Inst. H. Poincaré Anal. Non Linéaire 9 (1990) 201–218.

[11] M. Struwe, Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer-Verlag, 1996.

[12] S. Terracini, On positive entire solutions to a class of equations with a singular coefficient and critical exponent, Adv. Differential Equations 1 (1996) 241–264.

[13] G.M. Troianiello, Elliptic Differential Equations and Obstacle Problems, Plenum Press, 1987.

[14] X. Zhu, A perturbation result on positive entire solutions of a semilinear elliptic equation, J. Differential Equations 92 (1991) 163–178.